The normal distribution

Introduction

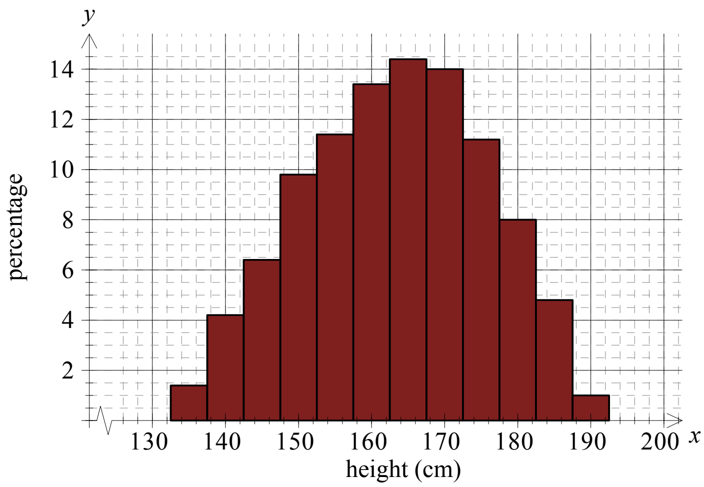
Here we look at the normal distribution. This is a continuous distribution that occurs naturally, for example the weights of babies at birth, the time taken to get to work every day, or the height of adults.

These variables all follow the same pattern. There will be some low values and some high values, but the majority of values will lie somewhere in the 'middle'.

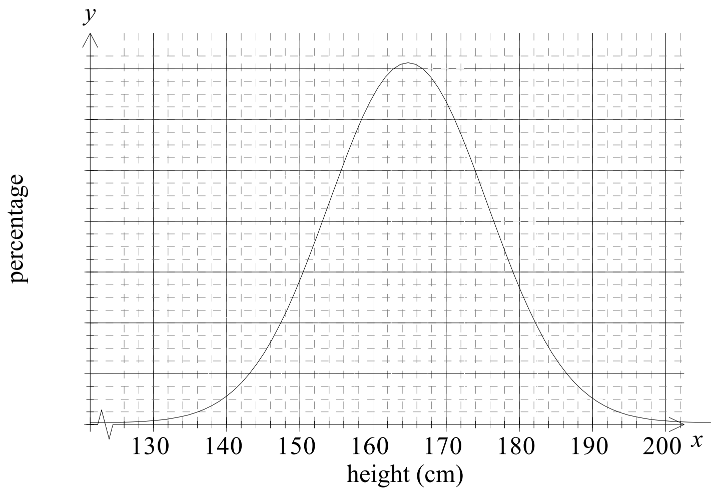
Features of the normal distribution

General features

If we were to plot a histogram of the heights of all female student actuaries, we would probably end up with a histogram like this:



As we sample a larger and larger group (and use smaller classes) we will approach the following graph:



The distribution is symmetrical, with most heights around the average of 165cm and fewer and fewer people as we approach the extremes. This symmetrical bell-shaped distribution is called the **normal distribution**. It occurs naturally in many other areas, for example: weights, IQ's, exam scores, and so on.

The PDF of the normal distribution

The normal distribution depends only on two parameters, μ and σ^2, which you will recognise as the mean and variance. So μ and σ^2 are the only parameters that appear in the probability density function (PDF):

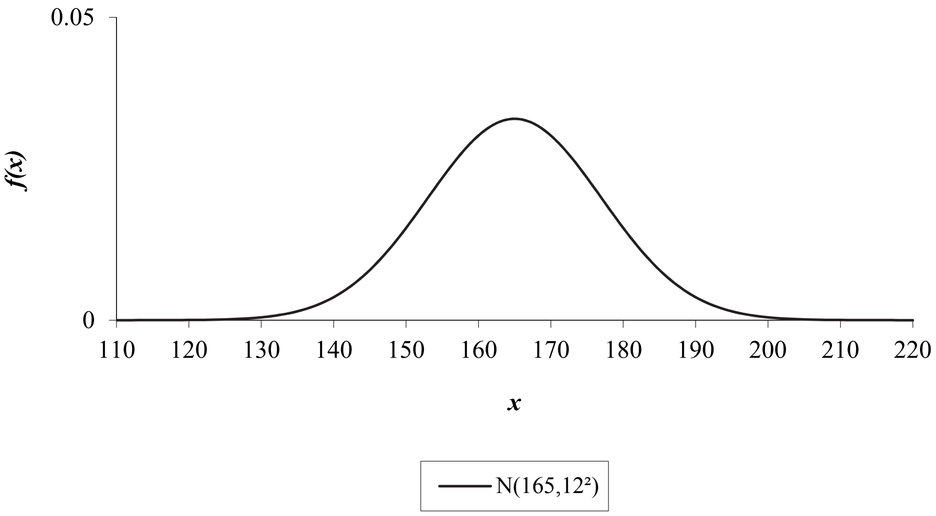
$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad -\infty < x < \infty$$

The shortcut way of writing *X* has a normal distribution with mean μ and variance σ^2 is:

$$X \sim N(\mu, \sigma^2)$$

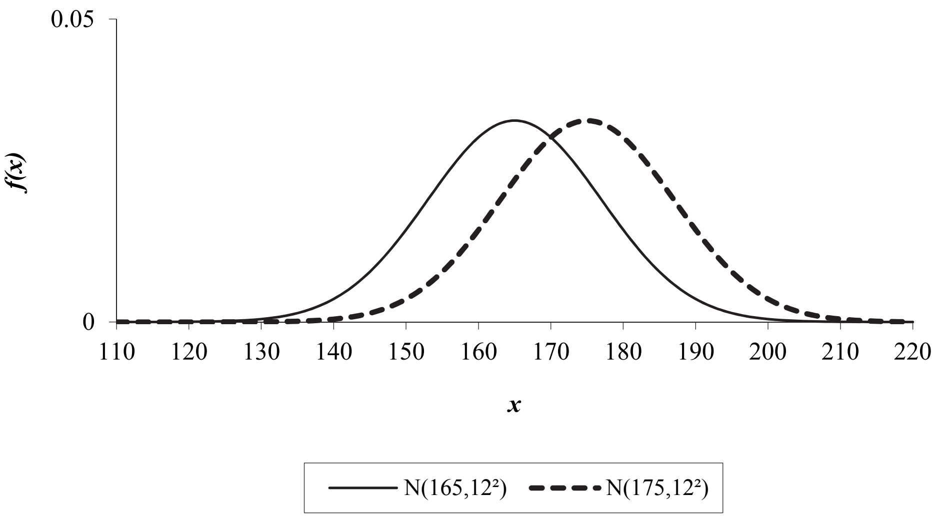
Now the PDF is grotty, but it does give the lovely symmetrical bell-shaped curve that we saw earlier. To get a better 'feel' for the shape of the PDF, we'll look at a number of normal distributions with different values of μ and σ^2.

Let's start with the distribution of the heights of our female actuaries, which had a mean of 165cm and a standard deviation of 12cm (so variance of 12^2 cm^2), *ie* $X \sim N(165, 12^2)$:



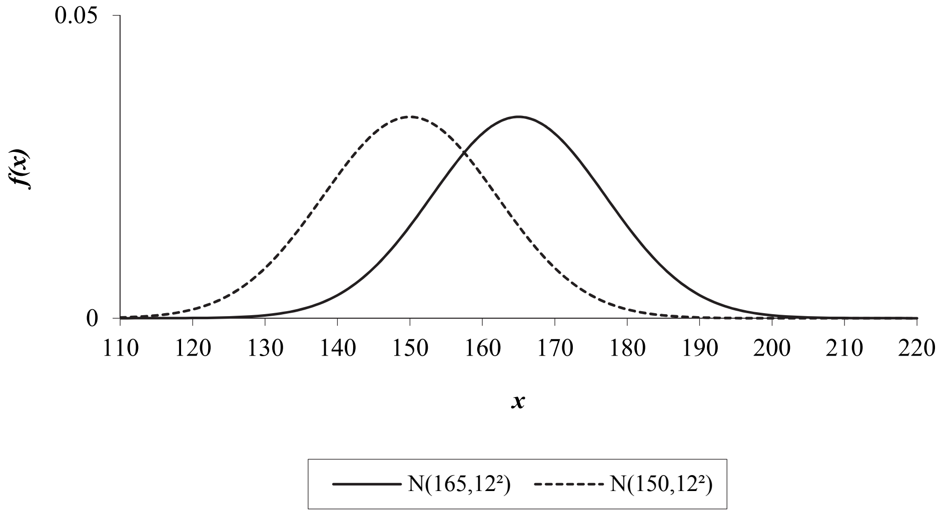
We can see that the PDF is greatest around the mean of 165cm and decreases as it moves further and further away from the mean.

So what happens if we increase the mean from 165cm to 175cm? Well, we would expect the female actuaries to be taller on average:



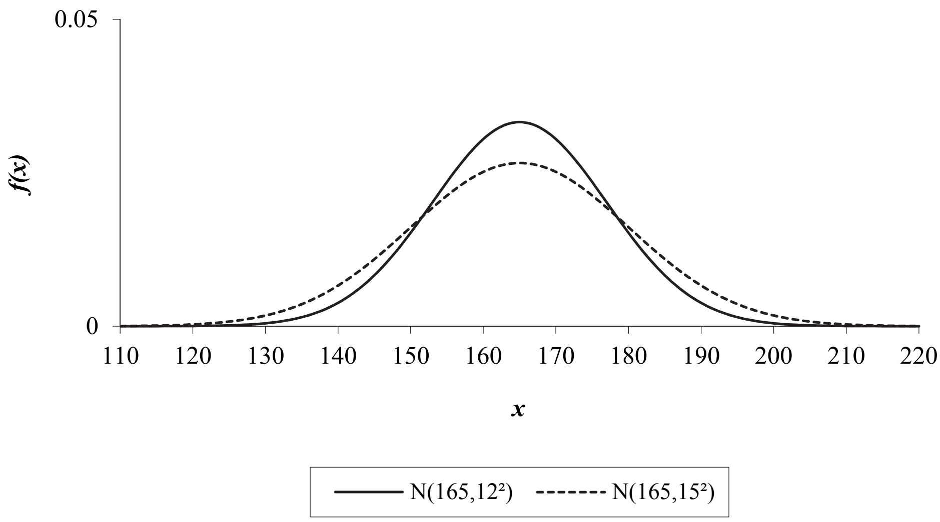
As you can see, the graph is simply shifted to the right, as all female actuaries are now 10 cm taller on average. Since we didn't alter the spread (the variance) the shape is still the same.

Similarly, if we were to reduce the mean from 165 cm to 150 cm we would see that the graph shifts to the left. All the female actuaries are 15 cm shorter on average. Again, the shape is the same, as we didn't change the variance.



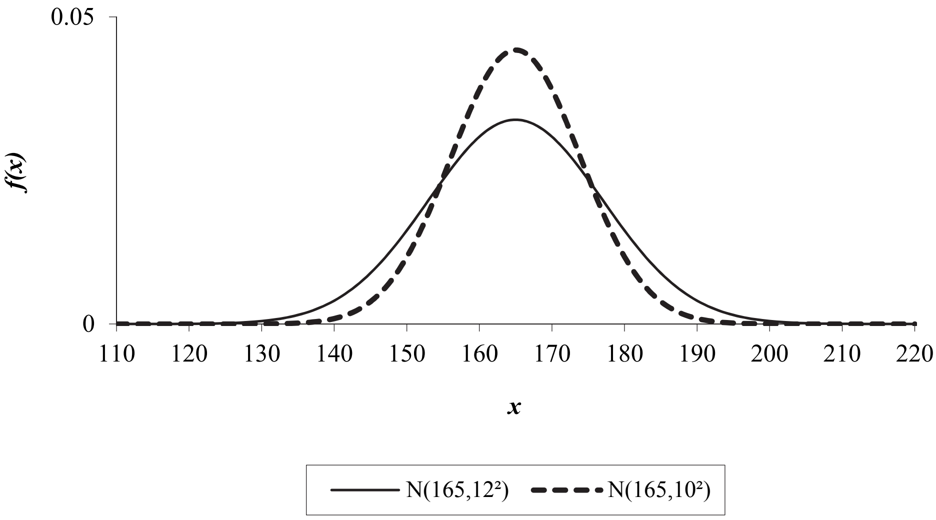
In summary, changing the mean, μ, simply moves the position of the normal PDF along the *x*-axis – it doesn't change the shape of the PDF.

So what happens if we increase the variance from 12^2 cm^2 to 15^2 cm^2? Well, we would expect the heights of the female actuaries to be more spread out:



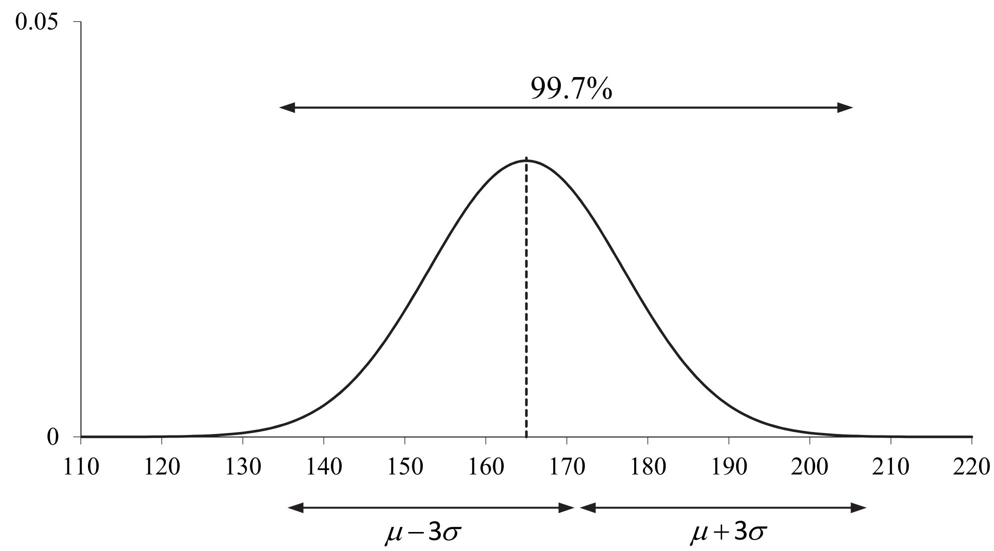
Here we can see that the heights of the actuaries are spread over more values than before. Consequently, there is a smaller percentage of people in the middle due to the greater variety in the heights.

If we now reduce the variance from 12^2 cm^2 to 10^2 cm^2, we should see a smaller range of heights. Consequently, there will be a larger percentage of people in the middle as more people have a similar height:



In summary, changing the variance, σ^2, alters the shape of the PDF of the normal distribution. A smaller variance 'squashes' it, whereas a larger variance 'stretches' it.

One last thing to note is that nearly all (in fact 99.7%) of the results are within 3 standard deviations of the mean. So for our female actuaries, with the $N(165,12^2)$ distribution, nearly all the heights are between $(165-3\times 12, 165+3\times 12)=(129,201)$:



Question 1.1

For each part sketch the two PDF graphs on the same diagram:

(i) $X \sim N(50,5^2)$ and $Y \sim N(50,10^2)$

(ii) $X \sim N(50,5^2)$ and $W \sim N(60,5^2)$.

Recall that one of the properties of a PDF is that the area under its graph equals to 1:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

We will not prove this result here as the proof is too complex.

Moments of the normal distribution

The moments are easily defined for $X \sim N(\mu,\sigma^2)$ as the mean is just μ and the variance is just σ^2.

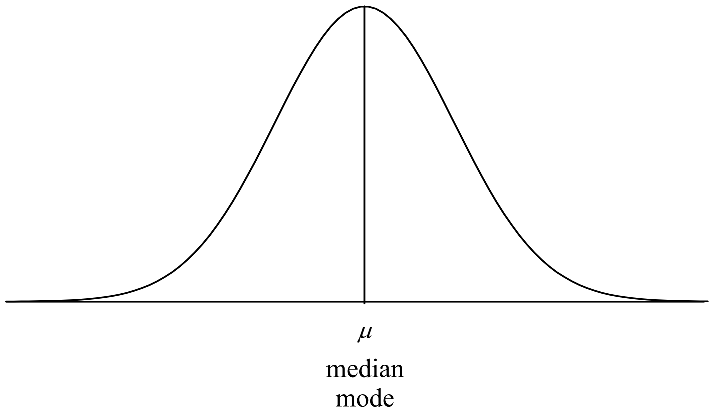
$$E(X) = \mu$$

$$\text{var}(X) = \sigma^2$$

Question 1.2

Determine $E(X^2)$ for $X \sim N(\mu,\sigma^2)$.

From the graph of the PDF of a normal distribution, it is clear that the mode is the same as the mean. Also, by symmetry the median is the same as the mean.



Probabilities of a normal distribution

To calculate probabilities for a continuous random variable we just integrate the PDF:

$$P(a < X < b) = \int_a^b f(x)\,dx$$

However, as mentioned earlier, integrating the PDF of a normal distribution is a nightmare, so we need another way. We could use statistical tables, but with so many combinations of μ and σ^2 which one do we tabulate? It turns out that we only need $N(0,1)$.

The standard normal distribution

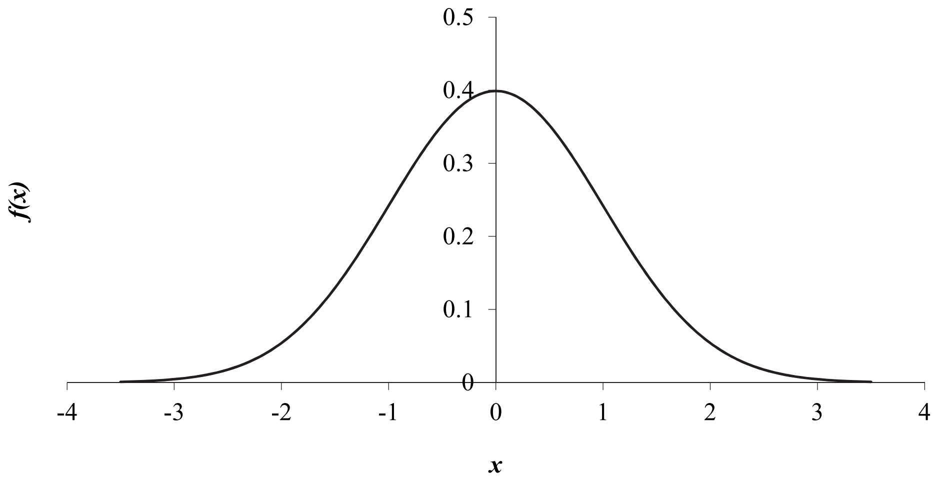
The **standard normal distribution** is a special case of the normal distribution with a mean of 0 and a variance of 1. We use the letter Z to stand for a random variable with a standard normal distribution:

$$Z \sim N(0,1)$$

The PDF of the standard normal distribution is obtained by simply substituting $\mu = 0$ and $\sigma^2 = 1$ into the normal PDF. This gives:

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \qquad -\infty < z < \infty$$

standard normal PDF is often denoted $\phi(z)$. It has the following shape:



We can see that the PDF is symmetrical about zero and that nearly all of the values lie within 3 standard deviations of the mean, *ie* between $(0-3\times1, 0+3\times1)=(-3,3)$.

The moments are:

$$E(Z)=0$$

$$\text{var}(Z)=1$$

The median is the same as the mean (by symmetry), as is the mode.

Probabilities of the standard normal distribution

The standard normal distribution is the only normal distribution that is tabulated. In this section, we will look at how to find probabilities using this table.

Simple probabilities

The cumulative distribution function of the standard normal is included in Appendix A. In the ACET exam you will be given the relevant values.

Recall that the cumulative density function, $F(x)$, is defined to be:

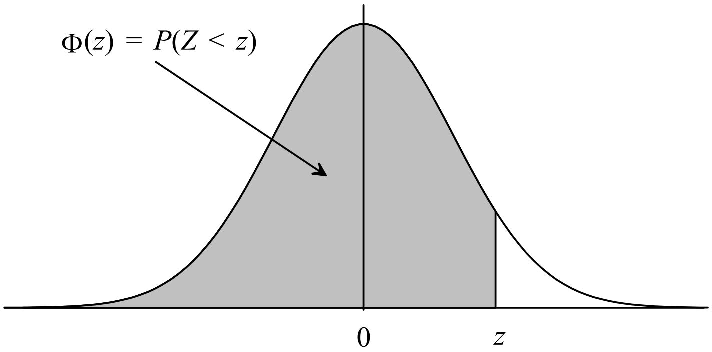
$$F(x) = P(X < x)$$

ie the probability that the random variable is less than *x*.

The standard normal cumulative density function is often denoted $\Phi(z)$:

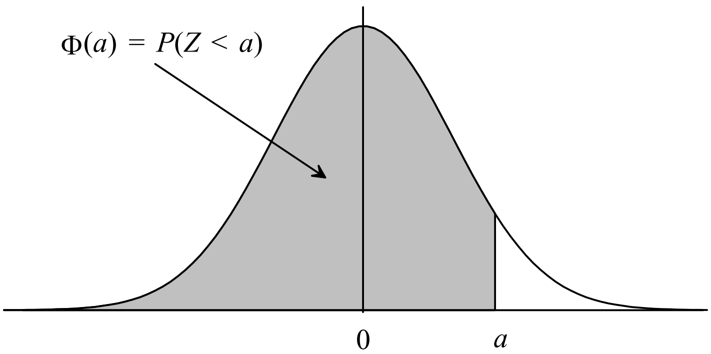
$$\Phi(z) = P(Z < z)$$

Notice how the table only gives 'less than' probabilities for positive values of *z*. Since the area under the curve represents the probability, this is shown as:



Calculating $P(Z < a)$

Firstly, we'll find the probability that *Z* (the standard normal random variable) is less than a positive number, say, *a*:



We simply read the values from the table. For example:

$$P(Z < 1.39) = 0.91774$$

1.39 is the value of x given (in bold) in the columns of the table in Appendix A.
The probability is given by $\Phi(x)$ *ie* the next column (which is not bold).

Question 1.3

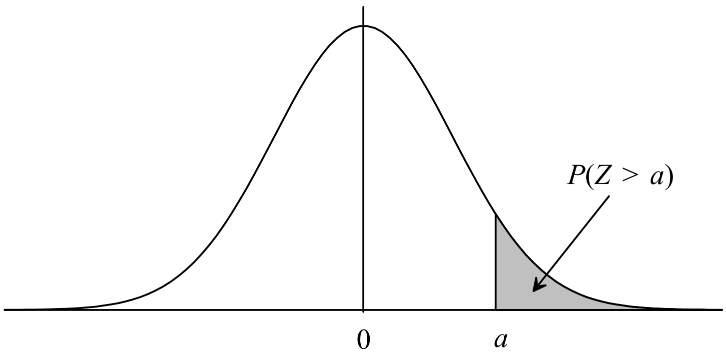
Find these probabilities:

(i) $P(Z<1.23)$

(ii) $P(Z<2.725)$.

Calculating $P(Z>a)$

We also need to find the probability that Z (the standard normal random variable) is more than a positive number, say, a:



The whole area under the graph (*ie* the total probability) is 1. This gives the following relationship:

$$P(Z<a)+P(Z>a)=1$$

Therefore:

$$P(Z>a)=1-P(Z<a)$$

So all we do is look up the 'less than' probability given in the standard normal tables and then subtract it from 1. For example:

$$P(Z>0.26)=1-P(Z<0.26)=1-0.60257=0.39743$$

Question 1.4

Find these probabilities:

(i) $P(Z > 2.17)$

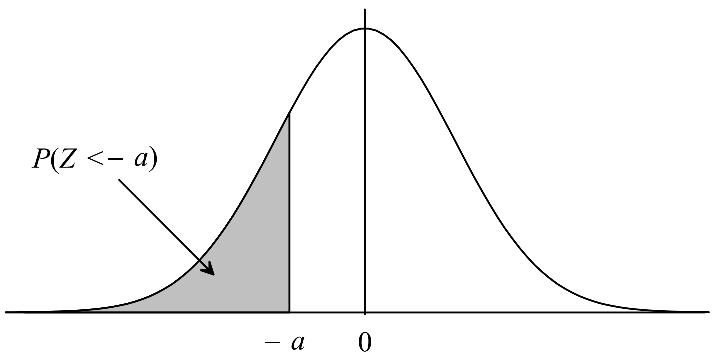
(ii) $P(Z > 0.08)$.

Using the notation $\Phi(z) = P(Z < z)$, we could write 'more than' probabilities as:

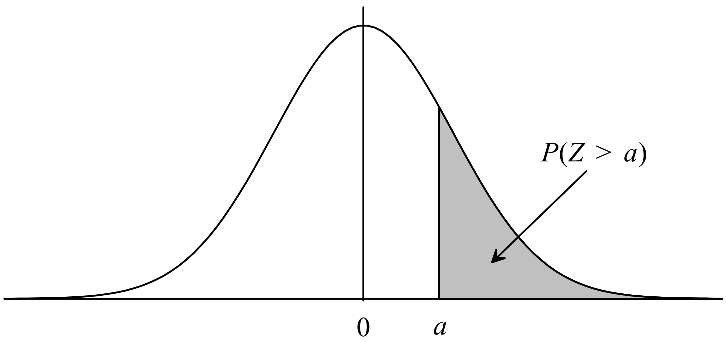
$$P(Z > a) = 1 - \Phi(a)$$

Calculating $P(Z < -a)$

Now we'll find the probability that Z (the standard normal random variable) is less than a negative number, say, $-a$:



problem is that negative values are not tabulated. So how do we do it? We use the fact that the normal distribution is symmetrical. By symmetry, the area shaded in the diagram above is *exactly* the same as the area shaded in the diagram below:



This area corresponds to the probability that Z (the standard normal random variable) is greater than the positive number a:

$$P(Z < -a) = P(Z > a)$$

We just worked out how to find 'more than' probabilities on the previous page. So:

$$P(Z < -a) = P(Z > a) = 1 - P(Z < a)$$

For example, to find:

$$P(Z < -3.08)$$

We first use symmetry:

$$P(Z < -3.08) = P(Z > 3.08)$$

We now use the method for calculating 'more than' probabilities:

$$P(Z > 3.08) = 1 - P(Z < 3.08) = 1 - 0.99896 = 0.00104$$

Perhaps the easiest way to remember the symmetry result (other than drawing a diagram) is to note that we swap the sign of the inequality (from 'less than' to 'more than') and we swap the sign of the number (from negative to positive). So in short we *'swap the sign and swap the sign'*.

Question 1.5

Find these probabilities:

(i) $P(Z < -1.50)$

(ii) $P(Z < -0.21)$

(iii) $P(Z < -2.05)$.

Using the notation $\Phi(z) = P(Z < z)$, we could write these probabilities as:

$$P(Z < -a) = \Phi(-a)$$

We can then write the relationship as:

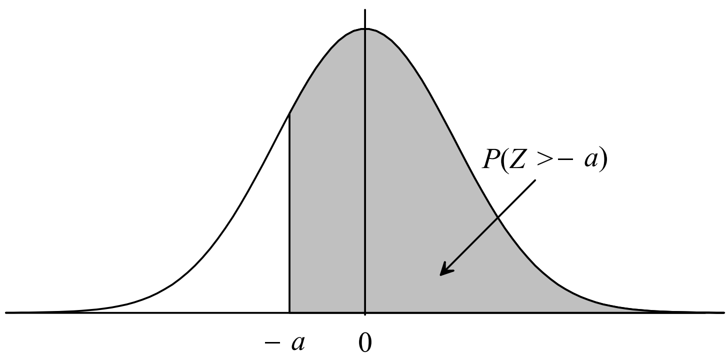
$$\Phi(-a) = 1 - \Phi(a)$$

ie:

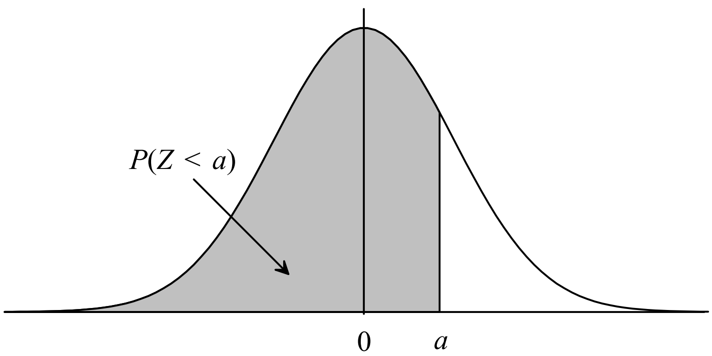
$$P(Z < -a) = 1 - \Phi(a)$$

Calculating $P(Z > -a)$

Finally we'll find the probability that Z (the standard normal random variable) is greater than a negative number, say, $-a$:



Again, since negative values are not tabulated we use the fact that the normal distribution is symmetrical. By symmetry, the area shaded in the diagram above is *exactly* the same as the area shaded in the diagram below:



This area corresponds to the probability that Z (the standard normal random variable) is less than the positive number a:

$$P(Z > -a) = P(Z < a)$$

We can just read this probability from the standard normal table in Appendix A.

For example, to find:

$$P(Z > -2.62)$$

First we use symmetry:

$$P(Z > -2.62) = P(Z < 2.62)$$

We now simply read this probability from the table in Appendix A:

$$P(Z < 2.62) = 0.99560$$

Once again we can remember the symmetry result by noting that we swap the sign of the inequality (from 'more than' to 'less than') and we swap the sign of the number (from negative to positive). So in short we *'swap the sign and swap the sign'*.

Question 1.6

Find these probabilities:

(i) $P(Z > -3.94)$

(ii) $P(Z > -0.73)$.

Using the notation $\Phi(z) = P(Z < z)$, we could write these probabilities as:

$$P(Z > -a) = \Phi(a)$$

Summary

We have now met all four of the probabilities that you could be asked:

Important result

$P(Z < a)$	read off the standard normal table.
$P(Z > a)$	$1 - P(Z < a)$ since area (and total probability) equals 1.
$P(Z < -a)$	$P(Z > a)$ using symmetry (*swap the sign and swap the sign*).
	$= 1 - P(Z < a)$ since area (and total probability) equals 1.
$P(Z > -a)$	$P(Z < a)$ using symmetry (*swap the sign and swap the sign*).

Since we will be using the normal distribution more than any other, it is vital that you can calculate these probabilities quickly. Take the time to learn the methods above. Try to do the following question *without* referring to the summary above.

Question 1.7

Find these probabilities:

(i) $P(Z > 3.6)$

(ii) $P(Z > -0.76)$

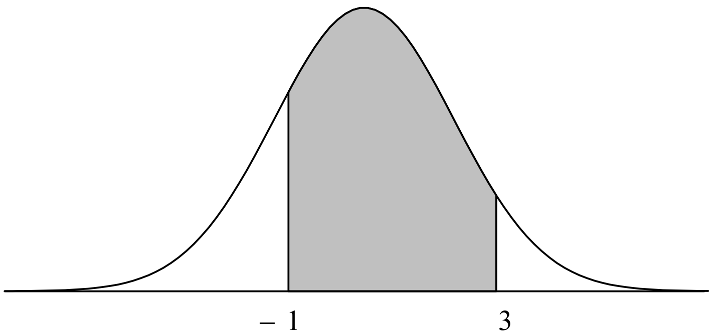
(iii) $P(Z < 1.98)$

(iv) $P(Z < -2.41)$.

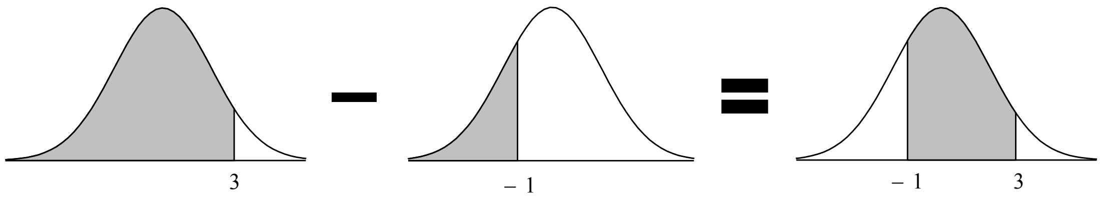
Compound probabilities

We can now calculate 'single' probabilities, like $P(Z < 3)$ or $P(Z > -1)$, so how can we use this to calculate 'compound' probabilities like $P(-1 < Z < 3)$?

The easiest way to see how to calculate these probabilities is to look at a diagram:



probability $P(-1<Z<3)$ is the shaded area between -1 and 3. How can we get this from 'single' probabilities? Imagine that the normal distribution is a piece of paper that we are cutting out and we want to just leave the shaded part. It can be made by starting with the $P(Z<3)$ part and subtracting the $P(Z<-1)$:



$P(Z<3) \qquad - \qquad P(Z<-1) \qquad = \qquad P(-1<Z<3)$

So, in general we have:

> ***Important result***
>
> $P(a<Z<b)=P(Z<b)-P(Z<a)$ for $a<b$

So to calculate a 'compound' probability, we first split it up into two 'single' probabilities. We then calculate each of those 'single' probabilities as before.

Completing our example, we get:

$$P(-1<Z<3)=P(Z<3)-P(Z<-1)$$

We can read $P(Z<3)$ directly from the standard normal table:

$$P(Z<3)=0.99865$$

To get $P(Z<-1)$, we need to use symmetry to get a positive value and then with a little rearranging we get $P(Z<1)$ which we can read from the standard normal table:

$$\begin{aligned} P(Z<-1) &= P(Z>1) \\ &= 1-P(Z<1) \\ &= 1-0.84134=0.15866 \end{aligned}$$

Now we substitute these two answers back into our original equation:

$$\begin{aligned} P(-1<Z<3) &= P(Z<3)-P(Z<-1) \\ &= 0.99865-0.15866 \\ &= 0.83999 \end{aligned}$$

These questions *are* common, so it is vital that you can calculate these probabilities quickly.

Question 1.8

Find these probabilities:

(i) $P(1.24<Z<2.19)$

(ii) $P(-0.92<Z<0.83)$

(iii) $P(-2.92<Z<-1.67)$.

Common Error:

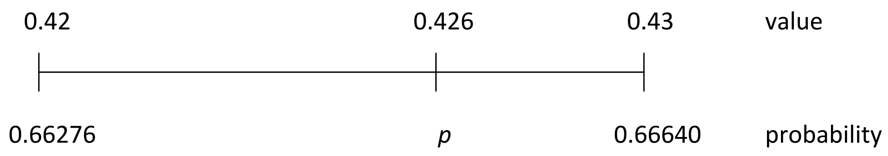
Many students calculate $P(a<Z<b)=P(Z<b)-P(Z>a)$ instead.

Probabilities involving interpolation

We will look at one last thing before we move on to how we can calculate probabilities for any normal distribution *other* than the standard normal. How do we calculate probabilities like $P(Z < 0.426)$?

The table in Appendix A only has $P(Z < 0.42)$ and $P(Z < 0.43)$ tabulated. So how do we get a probability for a value that is in between these? We can use **linear interpolation**.

Linear interpolation assumes that the probabilities increase *linearly* between values. This means that we can use proportions to find our 'in between' probability. Consider a number line:



The proportion of the 'length' that 0.426 is between 0.42 and 0.43:

$$\frac{0.426 - 0.42}{0.43 - 0.42} = 0.6$$

The proportion of the 'length' that the probability, *p*, is between 0.66278 and 0.66640:

$$\frac{p - 0.66276}{0.66640 - 0.66276}$$

For linear interpolation, we assume that the probabilities are spread out linearly between values. This means that the proportions will be equal. Hence:

$$\frac{p - 0.66276}{0.66640 - 0.66276} = 0.6$$

Rearranging this we get:

$$p = 0.66276 + 0.6 \times (0.66640 - 0.66276) = 0.66494$$

You may think 'is it worth it?' Well if we had just rounded 0.426 to 0.43, our answer would have been 66.64% instead of 66.49%. This is fairly serious.

So we need a quick way of getting to the last line. The first thing to notice is that the proportion 0.**6** is just the third decimal place of 0.43**6**. So had we been finding the probability for 1.28**4** we would have used a proportion of 0.**4**.

In general, we have:

$$p = (\text{start probability}) + 0.\cdots \times (\text{difference between probabilities})$$

proportion:
just use 3rd decimal
place (onwards)

We'll use this shortcut method to calculate a slightly harder probability:

$$P(Z < -1.793)$$

First we need to change it into a probability we can read from the standard normal table. Using the '*swap the sign, swap the sign*' symmetry rule and the 'more than' rule gives:

$$P(Z < -1.793) = P(Z > 1.793) = 1 - P(Z < 1.793)$$

Now we look up the probabilities each side of 1.793:

$$P(Z < 1.79) = 0.96327 \text{ and } P(Z < 1.80) = 0.96407$$

Now we use our linear interpolation rule:

$$P(Z < 1.79\mathbf{3}) = 0.96327 + 0.\mathbf{3} \times (0.96407 - 0.96327) = 0.96351$$

So we have:

$$P(Z < -1.793) = 1 - 0.96351 = 0.03649$$

Question 1.9

Calculate these probabilities using linear interpolation:

(i) $P(Z < 1.048)$

(ii) $P(Z > 0.271)$

(iii) $P(Z > -2.389)$

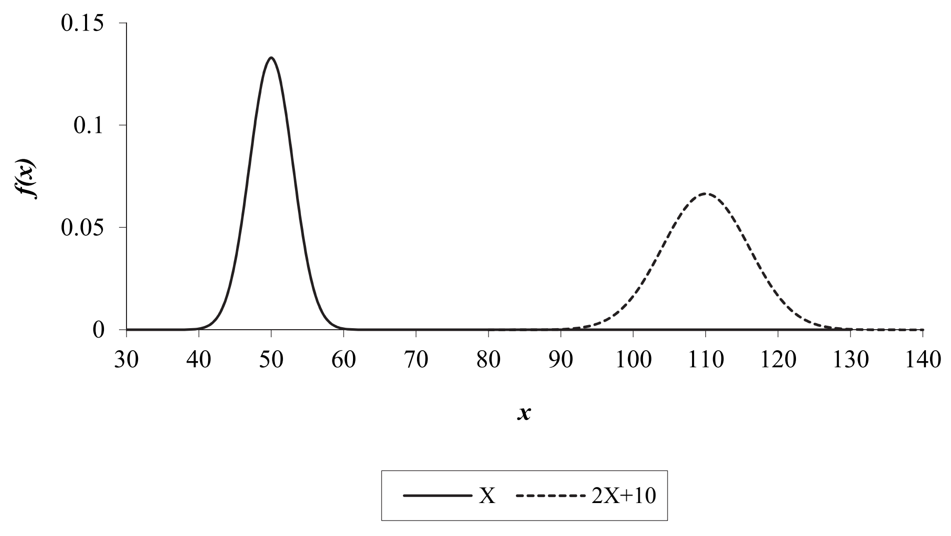
(iv) $P(-0.704 < Z < 0.897)$.

Probabilities for any normal distribution

So far we have only calculated probabilities for the standard normal distribution, $Z \sim N(0,1)$. Now we need to calculate probabilities for any other normal distribution.

Transforming normal distributions

We use the fact that a linear function of a normal random variable has a normal distribution. For example, if X has a normal distribution, say $X \sim N(50, 3^2)$, then $2X+10$ *also* has a normal distribution:



All we need to do now is to identify the new mean and variance of our transformed function.

First let's look at how the mean has changed. Recall the following result for *any* distribution:

$$E(aX+b)=aE(X)+b$$

ie if we multiply the random variable by *a* and then add *b*, the mean of the random variable is also multiplied by *a* with *b* added on.

Now we had $X \sim N(50,3^2)$ with $E(X)=50$, so $2X+10$ should have a mean of:

$$2E(X)+10=2\times 50+10=110$$

We can see on the diagram that $2X+10$ does indeed have a mean of 110.

Next let's look at how the variance has changed. Recall the following result for *any* distribution:

$$\text{var}\left(aX+b\right)=a^2\ \text{var}\left(X\right)$$

ie if we multiply the random variable by *a* and then add *b*, the variance of the random variable is multiplied by a^2.

Now we had $X \sim N(50,3^2)$ with $\text{var}\left(X\right)=3^2$, so $2X+10$ should have a variance of:

$$2^2\ \text{var}(X)=2^2\times 3^2=6^2$$

We can see this on the diagram (remembering that the majority of the values of the normal distribution are spread over 3 standard deviations each side of the mean). $2X+10$ is spread over $(110-3\times 6,110+3\times 6)=(92,128)$.

So in general, we have:

Important result

If $X \sim N(\mu,\sigma^2)$ then $aX+b \sim N(a\mu+b,a^2\sigma^2)$.

We will not formally prove this result here.

Question 1.10

If $X \sim N(100, 4^2)$, write down the distribution of:

(i) $3X + 5$

(ii) $½X - 20$

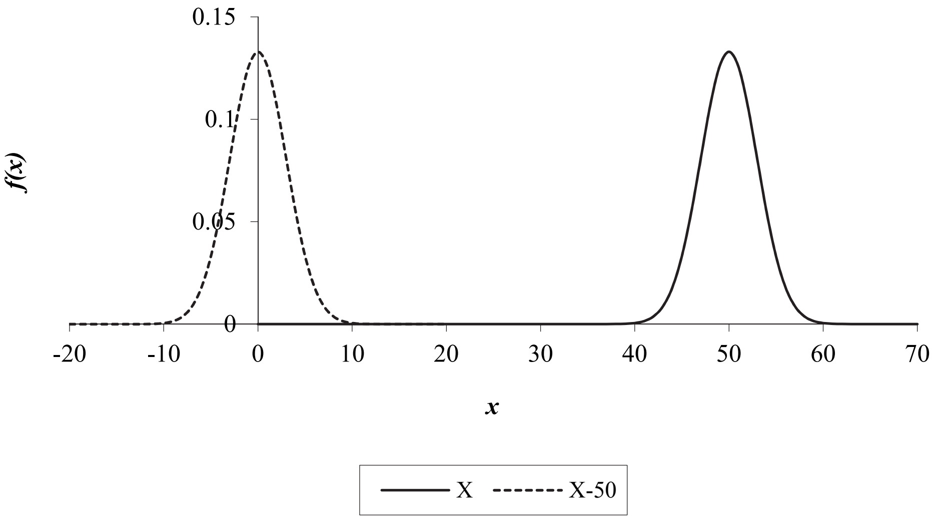
(iii) $¼(X - 100)$.

Standardising

We now know how to change one type of normal distribution into another normal distribution. So how does this help us find probabilities for any normal distribution?

What we are going to do is transform our normal distribution into the standard normal distribution. We can then look up the probabilities in the standard normal table in Appendix A. So how do we do it?

Let's take our normal distribution from before, $X \sim N(50, 3^2)$. We want to change this into the standard normal $Z \sim N(0,1)$. First we want to change the mean from 50 to 0. The easiest way to do this is to subtract 50:



We had $X \sim N(50,3^2)$ with $E(X)=50$, so $X-50$ has a mean of:

$$E(X-50)=E(X)-50=50-50=0$$

Next we want to change the variance from 3^2 to 1. Since only multiplying or dividing affects the variance, we need to divide the random variable by 3:

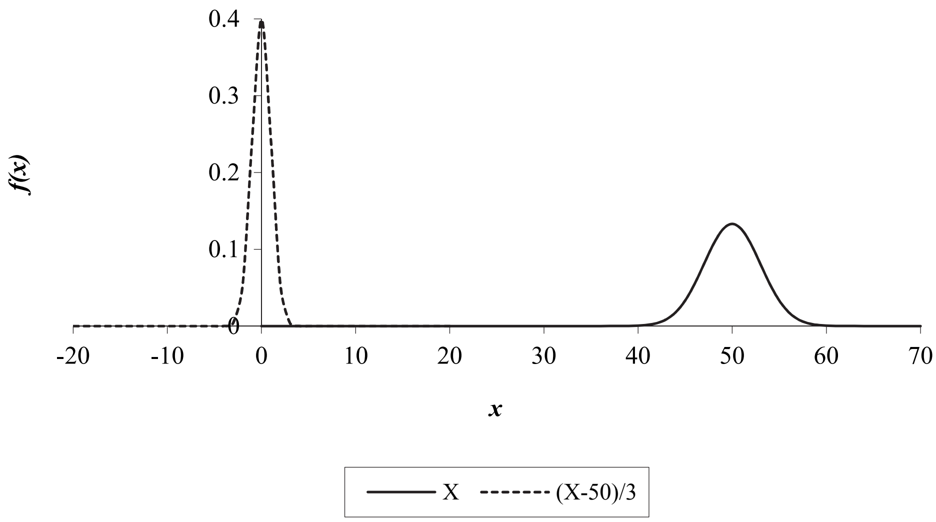
$$\text{var}(\tfrac{1}{3}X)=\tfrac{1}{3^2}\text{var}(X)=\tfrac{1}{3^2}3^2=1$$

Now do we divide by the 3 before or after subtracting the 50? If we did it before we would have a mean of:

$$E(\tfrac{1}{3}X-50)=\tfrac{1}{3}E(X)-50=\tfrac{1}{3}50-50=-33\tfrac{1}{3}$$

Whereas if we did it after we would have a mean of:

$$E[\tfrac{1}{3}(X-50)]=\tfrac{1}{3}E(X-50)=\tfrac{1}{3}[E(X)-50]=\tfrac{1}{3}(50-50)=0$$



Although it's squashed because of the scale, we can see that we have now transformed our normal distribution into the standard normal distribution. This process is called **standardising** the normal distribution.

How do we do this in general? Well for $X \sim N(50,3^2)$ we took away the mean, 50, first and *then* we divided it by 3 which was the standard deviation.

So for $X \sim N(\mu,\sigma^2)$ we would subtract the mean, μ, first and then divide by the standard deviation, σ, second:

$$\frac{X-\mu}{\sigma}$$

This gives us the standard normal distribution, $Z \sim N(0,1)$.

Standardising

If $X \sim N(\mu,\sigma^2)$ and $Z = \dfrac{X-\mu}{\sigma}$ then $Z \sim N(0,1)$.

Calculating probabilities for any normal distribution

We can now use this idea of subtracting the mean and dividing by the standard deviation to change the probability for *any* normal distribution into a probability for a standard normal distribution.

Taking our normal distribution of $X \sim N(50,3^2)$, let's find $P(X>56)$.

First we subtract the mean of 50 from *both* sides:

$$P(X>56) = P(X-50>6)$$

Next we divide *both* sides by the standard deviation of 3:

$$P(X>56) = P\left(\frac{X-50}{3}>2\right)$$

Now we use the fact that when we subtract the mean and divide by the standard deviation we get a standard normal distribution:

$$P(X>56) = P(Z>2)$$

We can now look this probability up in our standard normal table:

$$P(Z>2) = 1-P(Z<2) = 1-0.97725 = 0.02275$$

So we have:

$$P(X > 56) = 0.02275$$

This may seem a little longwinded at the moment (and more so when we need to use interpolation) but it does get quicker with a little practice. Usually we jump straight to:

$$P(X > 56) = P\left(Z > \frac{56-50}{3}\right) = P(Z > 2)$$

Because this is *the* method for calculating probabilities, we'll work through another example just to make sure it's totally clear:

If $X \sim N(15,25)$, calculate:

(i) $P(X < 18)$ (ii) $P(X > 11.8)$ (iii) $P(9.8 < X < 18.2)$

Working through each of these in turn:

(i) $P(X < 18) = P\left(Z < \frac{18-15}{\sqrt{25}}\right) = P(Z < 0.6) = 0.72575$

This value is read directly from the standard normal table.

(ii) $P(X > 11.8) = P\left(Z > \frac{11.8-15}{\sqrt{25}}\right) = P(Z > -0.64)$

Since we have a negative number we use '*swap the sign, swap the sign*', we can then read the resulting probability directly from the standard normal table:

$$P(X > 11.8) = P(Z > -0.64) = P(Z < 0.64) = 0.73891$$

(iii) We first need to split up this 'compound' probability:

$$P(9.8 < X < 18.2) = P(X < 18.2) - P(X < 9.8)$$

Now we standardise and calculate each part:

$$P(X < 18.2) = P\left(Z < \frac{18.2 - 15}{\sqrt{25}}\right) = P(Z < 0.64) = 0.73891$$

This value is read directly from the standard normal table.

$$P(X < 9.8) = P\left(Z < \frac{9.8 - 15}{\sqrt{25}}\right) = P(Z < -1.04)$$

Using the '*swap the sign, swap the sign*' rule and the fact that less than and more than probabilities sum to 1 we get:

$$\begin{aligned} P(Z < -1.04) &= P(Z > 1.04) \\ &= 1 - P(Z < 1.04) \\ &= 1 - 0.85083 = 0.14917 \end{aligned}$$

Hence:

$$P(9.8 < X < 18.2) = 0.73891 - 0.14917 = 0.58974$$

Question 1.11

If $X \sim N(100,16)$, calculate these probabilities using standardisation:

(i) $P(X > 110)$

(ii) $P(X > 87)$

(iii) $P(95 < X < 107)$.

Common Error:

Many students divide by the variance instead of the standard deviation.

This next question involves interpolation as well as standardising:

Question 1.12

The heights of male actuaries are normally distributed with mean 178 cm and variance 250 cm^2. Calculate the probability that a randomly chosen male actuary has height:

(i) less than 186 cm

(ii) more than 160 cm

(iii) between 150 cm and 190 cm.

Miscellaneous problems

This last section covers two types of question involving normal distribution probabilities.

Conditional probabilities

We can work out conditional probabilities using the conditional probability formula:

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

For example, to calculate $P(X < 3 \mid X < 6)$ where $X \sim N(5,4)$, we get:

$$P(X < 3 \mid X < 6) = \frac{P(X < 3 \text{ and } X < 6)}{P(X < 6)} = \frac{P(X < 3)}{P(X < 6)}$$

We now calculate these probabilities as before.

$$P(X<3)=P\left(Z<\frac{3-5}{\sqrt{4}}\right) \quad \text{standardising}$$

$$=P(Z<-1)$$

$$=P(Z>1) \quad \text{by symmetry}$$

$$=1-P(Z<1) \quad \text{using area}=1$$

$$=1-0.84134 \quad \text{from tables}$$

$$=0.15866$$

$$P(X<6)=P\left(Z<\frac{6-5}{\sqrt{4}}\right) \quad \text{standardising}$$

$$=P(Z<0.5)$$

$$=0.69146 \quad \text{from tables}$$

Hence:

$$P(X<3\,|\,X<6)=\frac{0.15866}{0.69146}=0.2295$$

Modulus

The modulus function, $|x|$, gives the positive value of x:

$$|x|=\begin{cases} x & x\geq 0 \\ -x & x<0 \end{cases}$$

So if we had:

$$|x|<3$$

this means that either:

$x<3$ if x was positive

or:

$-x<3 \quad \Rightarrow \quad x>-3$ if x was negative

So $|x|<3$ corresponds to the following range:

$$-3<x<3$$

If we are asked to calculate a probability involving a modulus, we rewrite it as a range:

$$P(|X|<5)=P(-5<X<5)$$

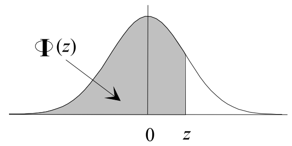
$$\begin{aligned}P(|X-3|<10)&=P(-10<X-3<10)\\&=P(-7<X<13)\end{aligned}$$

We can then calculate these in the normal (no pun intended) way.

Question 1.13

Calculate $P(|X-5|<2)$ where $X \sim N(6,1)$.

Appendix A: Probabilities for the standard normal distribution



z	Φ(z)	z	Φ(z)	z	Φ(z)	z	Φ(z)	z	Φ(z)	z	Φ(z)	z	Φ(z)	z	Φ(z)	z	Φ(z)	z	Φ(z)
0.00	0.50000	**0.40**	0.65542	**0.80**	0.78814	**1.20**	0.88493	**1.60**	0.94520	**2.00**	0.97725	**2.40**	0.99180	**2.80**	0.99744	**3.20**	0.99931	**3.60**	0.99984
0.01	0.50399	**0.41**	0.65910	**0.81**	0.79103	**1.21**	0.88686	**1.61**	0.94630	**2.01**	0.97778	**2.41**	0.99202	**2.81**	0.99752	**3.21**	0.99934	**3.61**	0.99985
0.02	0.50798	**0.42**	0.66276	**0.82**	0.79389	**1.22**	0.88877	**1.62**	0.94738	**2.02**	0.97831	**2.42**	0.99224	**2.82**	0.99760	**3.22**	0.99936	**3.62**	0.99985
0.03	0.51197	**0.43**	0.66640	**0.83**	0.79673	**1.23**	0.89065	**1.63**	0.94845	**2.03**	0.97882	**2.43**	0.99245	**2.83**	0.99767	**3.23**	0.99938	**3.63**	0.99986
0.04	0.51595	**0.44**	0.67003	**0.84**	0.79955	**1.24**	0.89251	**1.64**	0.94950	**2.04**	0.97932	**2.44**	0.99266	**2.84**	0.99774	**3.24**	0.99940	**3.64**	0.99986
0.05	0.51994	**0.45**	0.67364	**0.85**	0.80234	**1.25**	0.89435	**1.65**	0.95053	**2.05**	0.97982	**2.45**	0.99286	**2.85**	0.99781	**3.25**	0.99942	**3.65**	0.99987
0.06	0.52392	**0.46**	0.67724	**0.86**	0.80511	**1.26**	0.89617	**1.66**	0.95154	**2.06**	0.98030	**2.46**	0.99305	**2.86**	0.99788	**3.26**	0.99944	**3.66**	0.99987
0.07	0.52790	**0.47**	0.68082	**0.87**	0.80785	**1.27**	0.89796	**1.67**	0.95254	**2.07**	0.98077	**2.47**	0.99324	**2.87**	0.99795	**3.27**	0.99946	**3.67**	0.99988
0.08	0.53188	**0.48**	0.68439	**0.88**	0.81057	**1.28**	0.89973	**1.68**	0.95352	**2.08**	0.98124	**2.48**	0.99343	**2.88**	0.99801	**3.28**	0.99948	**3.68**	0.99988
0.09	0.53586	**0.49**	0.68793	**0.89**	0.81327	**1.29**	0.90147	**1.69**	0.95449	**2.09**	0.98169	**2.49**	0.99361	**2.89**	0.99807	**3.29**	0.99950	**3.69**	0.99989
0.10	0.53983	**0.50**	0.69146	**0.90**	0.81594	**1.30**	0.90320	**1.70**	0.95543	**2.10**	0.98214	**2.50**	0.99379	**2.90**	0.99813	**3.30**	0.99952	**3.70**	0.99989
0.11	0.54380	**0.51**	0.69497	**0.91**	0.81859	**1.31**	0.90490	**1.71**	0.95637	**2.11**	0.98257	**2.51**	0.99396	**2.91**	0.99819	**3.31**	0.99953	**3.71**	0.99990
0.12	0.54776	**0.52**	0.69847	**0.92**	0.82121	**1.32**	0.90658	**1.72**	0.95728	**2.12**	0.98300	**2.52**	0.99413	**2.92**	0.99825	**3.32**	0.99955	**3.72**	0.99990
0.13	0.55172	**0.53**	0.70194	**0.93**	0.82381	**1.33**	0.90824	**1.73**	0.95818	**2.13**	0.98341	**2.53**	0.99430	**2.93**	0.99831	**3.33**	0.99957	**3.73**	0.99990
0.14	0.55567	**0.54**	0.70540	**0.94**	0.82639	**1.34**	0.90988	**1.74**	0.95907	**2.14**	0.98382	**2.54**	0.99446	**2.94**	0.99836	**3.34**	0.99958	**3.74**	0.99991
0.15	0.55962	**0.55**	0.70884	**0.95**	0.82894	**1.35**	0.91149	**1.75**	0.95994	**2.15**	0.98422	**2.55**	0.99461	**2.95**	0.99841	**3.35**	0.99960	**3.75**	0.99991
0.16	0.56356	**0.56**	0.71226	**0.96**	0.83147	**1.36**	0.91308	**1.76**	0.96080	**2.16**	0.98461	**2.56**	0.99477	**2.96**	0.99846	**3.36**	0.99961	**3.76**	0.99992
0.17	0.56749	**0.57**	0.71566	**0.97**	0.83398	**1.37**	0.91466	**1.77**	0.96164	**2.17**	0.98500	**2.57**	0.99492	**2.97**	0.99851	**3.37**	0.99962	**3.77**	0.99992
0.18	0.57142	**0.58**	0.71904	**0.98**	0.83646	**1.38**	0.91621	**1.78**	0.96246	**2.18**	0.98537	**2.58**	0.99506	**2.98**	0.99856	**3.38**	0.99964	**3.78**	0.99992
0.19	0.57535	**0.59**	0.72240	**0.99**	0.83891	**1.39**	0.91774	**1.79**	0.96327	**2.19**	0.98574	**2.59**	0.99520	**2.99**	0.99861	**3.39**	0.99965	**3.79**	0.99992
0.20	0.57926	**0.60**	0.72575	**1.00**	0.84134	**1.40**	0.91924	**1.80**	0.96407	**2.20**	0.98610	**2.60**	0.99534	**3.00**	0.99865	**3.40**	0.99966	**3.80**	0.99993
0.21	0.58317	**0.61**	0.72907	**1.01**	0.84375	**1.41**	0.92073	**1.81**	0.96485	**2.21**	0.98645	**2.61**	0.99547	**3.01**	0.99869	**3.41**	0.99968	**3.81**	0.99993
0.22	0.58706	**0.62**	0.73237	**1.02**	0.84614	**1.42**	0.92220	**1.82**	0.96562	**2.22**	0.98679	**2.62**	0.99560	**3.02**	0.99874	**3.42**	0.99969	**3.82**	0.99993
0.23	0.59095	**0.63**	0.73565	**1.03**	0.84849	**1.43**	0.92364	**1.83**	0.96638	**2.23**	0.98713	**2.63**	0.99573	**3.03**	0.99878	**3.43**	0.99970	**3.83**	0.99994
0.24	0.59483	**0.64**	0.73891	**1.04**	0.85083	**1.44**	0.92507	**1.84**	0.96712	**2.24**	0.98745	**2.64**	0.99585	**3.04**	0.99882	**3.44**	0.99971	**3.84**	0.99994
0.25	0.59871	**0.65**	0.74215	**1.05**	0.85314	**1.45**	0.92647	**1.85**	0.96784	**2.25**	0.98778	**2.65**	0.99598	**3.05**	0.99886	**3.45**	0.99972	**3.85**	0.99994
0.26	0.60257	**0.66**	0.74537	**1.06**	0.85543	**1.46**	0.92785	**1.86**	0.96856	**2.26**	0.98809	**2.66**	0.99609	**3.06**	0.99889	**3.46**	0.99973	**3.86**	0.99994
0.27	0.60642	**0.67**	0.74857	**1.07**	0.85769	**1.47**	0.92922	**1.87**	0.96926	**2.27**	0.98840	**2.67**	0.99621	**3.07**	0.99893	**3.47**	0.99974	**3.87**	0.99995
0.28	0.61026	**0.68**	0.75175	**1.08**	0.85993	**1.48**	0.93056	**1.88**	0.96995	**2.28**	0.98870	**2.68**	0.99632	**3.08**	0.99896	**3.48**	0.99975	**3.88**	0.99995
0.29	0.61409	**0.69**	0.75490	**1.09**	0.86214	**1.49**	0.93189	**1.89**	0.97062	**2.29**	0.98899	**2.69**	0.99643	**3.09**	0.99900	**3.49**	0.99976	**3.89**	0.99995
0.30	0.61791	**0.70**	0.75804	**1.10**	0.86433	**1.50**	0.93319	**1.90**	0.97128	**2.30**	0.98928	**2.70**	0.99653	**3.10**	0.99903	**3.50**	0.99977	**3.90**	0.99995
0.31	0.62172	**0.71**	0.76115	**1.11**	0.86650	**1.51**	0.93448	**1.91**	0.97193	**2.31**	0.98956	**2.71**	0.99664	**3.11**	0.99906	**3.51**	0.99978	**3.91**	0.99995
0.32	0.62552	**0.72**	0.76424	**1.12**	0.86864	**1.52**	0.93574	**1.92**	0.97257	**2.32**	0.98983	**2.72**	0.99674	**3.12**	0.99910	**3.52**	0.99978	**3.92**	0.99996
0.33	0.62930	**0.73**	0.76730	**1.13**	0.87076	**1.53**	0.93699	**1.93**	0.97320	**2.33**	0.99010	**2.73**	0.99683	**3.13**	0.99913	**3.53**	0.99979	**3.93**	0.99996
0.34	0.63307	**0.74**	0.77035	**1.14**	0.87286	**1.54**	0.93822	**1.94**	0.97381	**2.34**	0.99036	**2.74**	0.99693	**3.14**	0.99916	**3.54**	0.99980	**3.94**	0.99996
0.35	0.63683	**0.75**	0.77337	**1.15**	0.87493	**1.55**	0.93943	**1.95**	0.97441	**2.35**	0.99061	**2.75**	0.99702	**3.15**	0.99918	**3.55**	0.99981	**3.95**	0.99996
0.36	0.64058	**0.76**	0.77637	**1.16**	0.87698	**1.56**	0.94062	**1.96**	0.97500	**2.36**	0.99086	**2.76**	0.99711	**3.16**	0.99921	**3.56**	0.99981	**3.96**	0.99996
0.37	0.64431	**0.77**	0.77935	**1.17**	0.87900	**1.57**	0.94179	**1.97**	0.97558	**2.37**	0.99111	**2.77**	0.99720	**3.17**	0.99924	**3.57**	0.99982	**3.97**	0.99996
0.38	0.64803	**0.78**	0.78230	**1.18**	0.88100	**1.58**	0.94295	**1.98**	0.97615	**2.38**	0.99134	**2.78**	0.99728	**3.18**	0.99926	**3.58**	0.99983	**3.98**	0.99997
0.39	0.65173	**0.79**	0.78524	**1.19**	0.88298	**1.59**	0.94408	**1.99**	0.97670	**2.39**	0.99158	**2.79**	0.99736	**3.19**	0.99929	**3.59**	0.99983	**3.99**	0.99997
0.40	0.65542	**0.80**	0.78814	**1.20**	0.88493	**1.60**	0.94520	**2.00**	0.97725	**2.40**	0.99180	**2.80**	0.99744	**3.20**	0.99931	**3.60**	0.99984	**4.00**	0.99997

This page has been left blank so that you can pull out the normal tables for reference.

Extra practice questions

P1.1 Sketch a graph of $N(100,100)$.

P1.2 Find:

(i) $P(Z>1.6)$

(ii) $P(Z<2.84)$

(iii) $P(Z<-0.42)$

(iv) $P(Z>-2.61)$

(v) $P(-1.69<Z<0.52)$

(vi) $P(-3.05<Z<-1.7)$.

P1.3 Calculate the following using interpolation:

(i) $P(Z<0.371)$

(ii) $P(Z>2.598)$

(iii) $P(Z>-0.904)$

(iv) $P(Z<-2.319)$

(v) $P(1.572<Z<3.087)$

(vi) $P(-1.382<Z<0.493)$.

P1.4 Calculate the following, using interpolation:

(i) $P(X > 117)$ where $X \sim N(128, 8^2)$

(ii) $P(Y < 30)$ where $Y \sim N(23, 5^2)$

(iii) $P(W > 297)$ where $W \sim N(285, 34)$

(iv) $P(V < 90)$ where $V \sim N(96, 12)$

(v) $P(3 < T < 8)$ where $T \sim N(5, 2)$.

P1.5 Given that $X \sim N(17, 5)$, evaluate:

(i) $P(|X - 17| < 2)$

(ii) $P(X < 15 | X < 16)$.

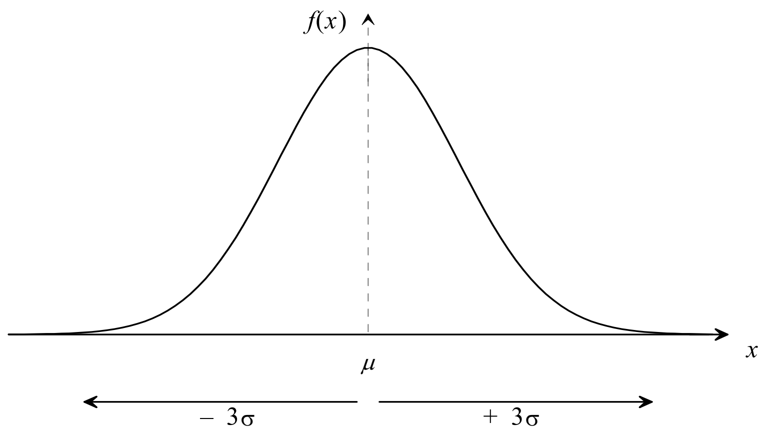
P1.6 The sizes of claims, which arise under policies of a certain type, are normally distributed with mean $\mu = £4{,}000$ and standard deviation $\sigma = £600$. The size of a particular claim is known to be greater than £3,400.

What is the probability that this claim size is greater than £4,000, the average of all claims sizes? [3]

Summary

The normal distribution

If X has a **normal distribution** with mean μ and variance σ^2 then we write $X \sim N(\mu, \sigma^2)$. This distribution occurs naturally and has a symmetrical, bell-shaped PDF:



The PDF for $X \sim N(\mu, \sigma^2)$ is:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad -\infty < x < \infty$$

The moments are:

$$E(X) = \mu$$

$$\text{var}(X) = \sigma^2$$

The median is the same as the mean (by symmetry) as is the mode.

Probabilities can only be found by **standardising** the normal distribution (*ie* transforming it into a standard normal distribution) using:

$$Z = \frac{X - \mu}{\sigma}$$

We then use the standard normal distribution tables.

standard normal distribution

This is a normal distribution with mean 0 and variance 1. We write:

$$Z \sim N(0,1)$$

It has PDF:

$$\phi(z) = f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

The cumulative density function $\Phi(z) = P(Z < z)$ is given in the table in Appendix A. We can use this to calculate probabilities as follows:

$P(Z < a)$ read off the tables

$P(Z > a) = 1 - P(Z < a)$ using area (and total probability) equals 1

$P(Z < -a) = P(Z > a)$ using symmetry

$= 1 - P(Z < a)$ using area (and total probability) equals 1

$P(Z > -a) = P(Z < a)$ using symmetry

'Compound' probabilities can be found using:

$$P(a < X < b) = P(X < b) - P(X < a)$$

Probabilities involving a modulus can be rewritten as a range:

$$P(|X| < a) = P(-a < X < a)$$

Linear interpolation

It is expected that standard normal values are given to *at least* 3 decimal places and a more accurate answer is obtained using linear interpolation between the tabulated values:

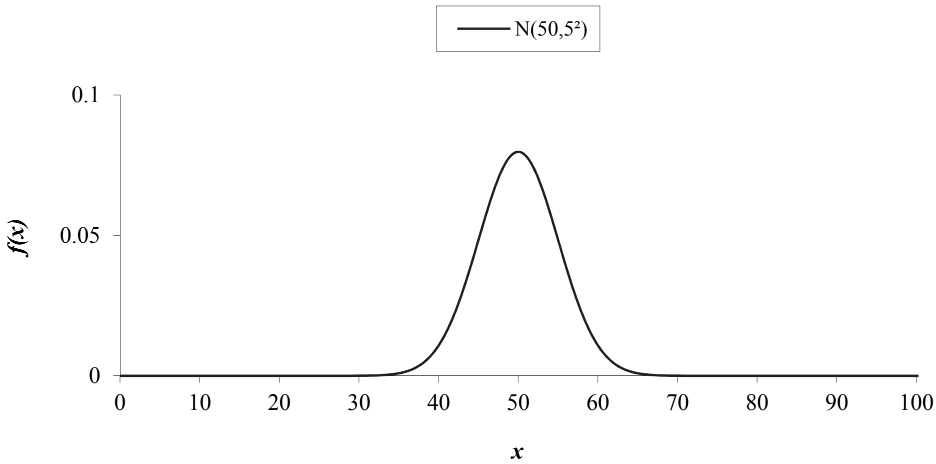
$$p = (\text{start probability}) + 0.\cdots \times (\text{difference between probabilities})$$

proportion:
just use 3rd decimal place (onwards)

Solutions

Solution 1.1

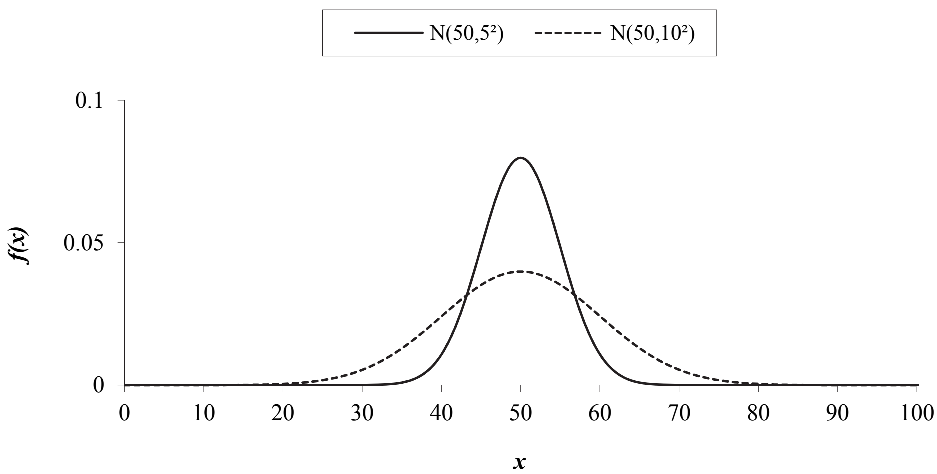
(i) The graph for $X \sim N(50,5^2)$ is:



The majority of the values lie between $(50-3\times5, 50+3\times5)=(35,65)$.

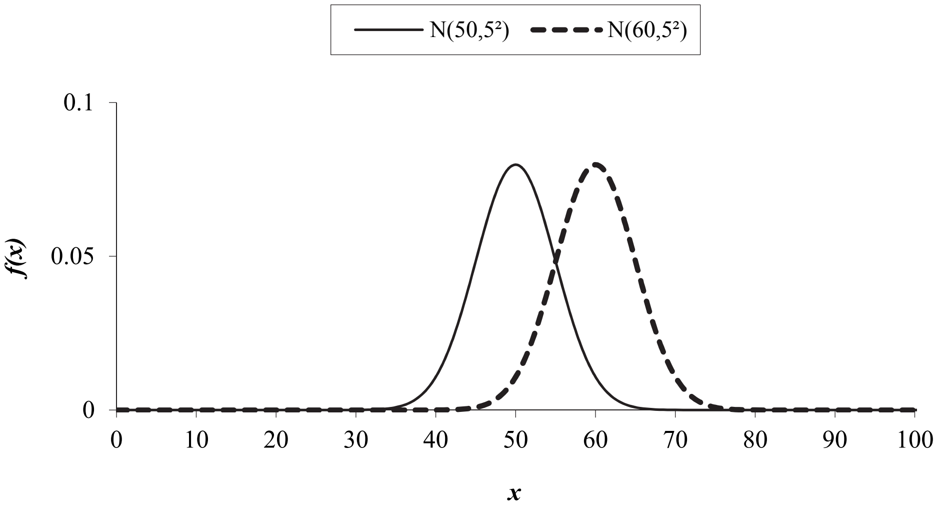
The height can be found by $f(50)=\dfrac{1}{\sqrt{2\pi5^2}}e^0=0.080$.

Now $Y \sim N(50,10^2)$ has the same mean but a larger variance, so the PDF will have the same position/centre but will be more spread out.



The majority of the values lie between $(50-3\times10, 50+3\times10)=(20,80)$. Since it's twice as spread out, it should be half as high.

(ii) Compared to $N(50,5^2)$, the mean of $N(60,5^2)$ has increased by 10 but the variance is the same, so the PDF will have the same shape/spread but will be shifted 10 to the right.



Solution 1.2

Since:

$$\text{var}(X) = E(X^2) - E^2(X)$$

This gives:

$$\begin{aligned} E(X^2) &= \text{var}(X) + E^2(X) \\ &= \sigma^2 + \mu^2 \end{aligned}$$

Solution 1.3

(i) Reading from the standard normal table:

$$P(Z<1.23)=0.89065$$

(ii) 2.725 is halfway between 2.72 and 2.73. Reading from the standard normal table:

$$P(Z<2.72)=0.99674$$

$$P(Z<2.73)=0.99683$$

Since 2.725 is halfway between these it will be:

$$P(Z<2.725)=\frac{0.99674+0.99683}{2}=0.996785$$

Solution 1.4

(i) Using the fact that the area under the graph is 1, we have:

$$P(Z>2.17)=1-P(Z<2.17)$$

Now reading $P(Z<2.17)$ from the standard normal table:

$$P(Z>2.17)=1-0.98500=0.01500$$

(ii) Similarly:

$$\begin{aligned} P(Z>0.08) &= 1-P(Z<0.08) && \text{using area}=1 \\ &= 1-0.53188 && \text{from tables} \\ &= 0.46812 \end{aligned}$$

Solution 1.5

(i) Using our '*swap the sign, swap the sign*' symmetry result:

$$P(Z<-1.50)=P(Z>1.50)$$

Now using the fact that the area under the graph is 1:

$$P(Z<-1.50)=P(Z>1.50)=1-P(Z<1.50)$$

Reading the value of $P(Z<1.50)$ from the standard normal table:

$$P(Z<-1.50)=1-0.93319=0.06681$$

(ii) Similarly,

$$\begin{aligned} P(Z<-0.21)&=P(Z>0.21) && \text{by symmetry} \\ &=1-P(Z<0.21) && \text{using area}=1 \\ &=1-0.58317 && \text{from tables} \\ &=0.41683 \end{aligned}$$

(iii) We get:

$$\begin{aligned} P(Z<-2.05)&=P(Z>2.05) && \text{by symmetry} \\ &=1-P(Z<2.05) && \text{using area}=1 \\ &=1-0.97982 && \text{from tables} \\ &=0.02018 \end{aligned}$$

Solution 1.6

(i) Using our '*swap the sign, swap the sign*' symmetry result:

$$P(Z > -3.94) = P(Z < 3.94)$$

We can read this value directly from the standard normal table:

$$P(Z > -3.94) = P(Z < 3.94) = 0.99996$$

(ii) Similarly,

$$P(Z > -0.73) = P(Z < 0.73) \quad \text{by symmetry}$$
$$= 0.76730 \quad \text{from tables}$$

Solution 1.7

(i) $P(Z > 3.6) = 1 - P(Z < 3.6)$ using area $= 1$

$= 1 - 0.99984$ from tables

$= 0.00016$

(ii) $P(Z > -0.76) = P(Z < 0.76)$ by symmetry

$= 0.77637$ from tables

(iii) $P(Z < 1.98) = 0.97615$ from tables

(iv) $P(Z < -2.41) = P(Z > 2.41)$ by symmetry

$= 1 - P(Z < 2.41)$ using area $= 1$

$= 1 - 0.99202$ from tables

$= 0.00798$

Solution 1.8

(i) Splitting up the 'compound' probability:

$$P(1.24 < Z < 2.19) = P(Z < 2.19) - P(Z < 1.24)$$

Both of these probabilities can be read directly from the standard normal table:

$$P(1.24 < Z < 2.19) = 0.98574 - 0.89251 = 0.09323$$

(ii) Similarly:

$$P(-0.92 < Z < 0.83) = P(Z < 0.83) - P(Z < -0.92)$$

Now:

$$P(Z < 0.83) = 0.79673 \qquad \text{from tables}$$

$$\begin{aligned} P(Z < -0.92) &= P(Z > 0.92) && \text{by symmetry} \\ &= 1 - P(Z < 0.92) && \text{using area} = 1 \\ &= 1 - 0.82121 && \text{from tables} \\ &= 0.17879 \end{aligned}$$

So this gives:

$$P(-0.92 < Z < 0.83) = 0.79673 - 0.17879 = 0.61794$$

(iii) Splitting up the 'compound' probability:

$$P(-2.92<Z<-1.67)=P(Z<-1.67)-P(Z<-2.92)$$

Now:

$$\begin{aligned} P(Z<-1.67) &= P(Z>1.67) && \text{by symmetry} \\ &= 1-P(Z<1.67) && \text{using area} = 1 \\ &= 1-0.95254 && \text{from tables} \\ &= 0.04746 \end{aligned}$$

$$\begin{aligned} P(Z<-2.92) &= P(Z>2.92) && \text{by symmetry} \\ &= 1-P(Z<2.92) && \text{using area} = 1 \\ &= 1-0.99825 && \text{from tables} \\ &= 0.00175 \end{aligned}$$

This gives:

$$P(-2.92<Z<-1.67)=0.04746-0.00175=0.04571$$

Solution 1.9

(i) To calculate $P(Z<1.048)$ we first look up the tabulated probabilities either side:

$$P(Z<1.04)=0.85083$$
$$P(Z<1.05)=0.85314$$

Using linear interpolation:

$$P(Z<1.04\mathbf{8})=0.85083+0.\mathbf{8}\times(0.85314-0.85083)=0.85268$$

Note that this figure is given to only 5 DP as the tables are only accurate to 5DP.

(ii) Now:

$$P(Z>0.271)=1-P(Z<0.271) \qquad \text{using area}=1$$

The tabulated probabilities either side of 0.271 are:

$$P(Z<0.27)=0.60642$$
$$P(Z<0.28)=0.61026$$

Using linear interpolation:

$$P(Z<0.27\mathbf{1})=0.60642+0.\mathbf{1}\times(0.61026-0.60642)=0.60680$$

So:

$$P(Z>0.271)=1-0.60680=0.39320$$

(iii) Now:

$$P(Z > -2.389) = P(Z < 2.389) \qquad \text{by symmetry}$$

The tabulated probabilities either side of 2.389 are:

$$P(Z < 2.38) = 0.99134$$

$$P(Z < 2.39) = 0.99158$$

Using linear interpolation:

$$P(Z < 2.38\mathbf{9}) = 0.99134 + 0.\mathbf{9} \times (0.99158 - 0.99134) = 0.99156$$

So:

$$P(Z > -2.389) = 0.99156$$

(iv) First we split up the 'compound' probability:

$$P(-0.704 < Z < 0.897) = P(Z < 0.897) - P(Z < -0.704)$$

For $P(Z < 0.897)$ the tabulated probabilities either side are:

$$P(Z < 0.89) = 0.81327$$

$$P(Z < 0.90) = 0.81594$$

Using linear interpolation:

$$P(Z < 0.89\mathbf{7}) = 0.81327 + 0.\mathbf{7} \times (0.81594 - 0.81327) = 0.81514$$

Now:

$$\begin{aligned} P(Z < -0.704) &= P(Z > 0.704) && \text{by symmetry} \\ &= 1 - P(Z < 0.704) && \text{using area} = 1 \end{aligned}$$

tabulated probabilities either side of $P(Z<$

$$P(Z<0.70)=0.75804$$

$$P(Z<0.71)=0.76115$$

Using linear interpolation:

$$P(Z<0.70\mathbf{4})=0.75804+0.\mathbf{4}\times(0.76115-0.75804)=0.75928$$

This gives:

$$P(Z<-0.704)=1-0.75928=0.24072$$

Hence:

$$P(-0.704<Z<0.897)=0.81514-0.24072=0.57442$$

Solution 1.10

(i) $3X+5 \sim N(3\times100+5, 3^2\times4^2)=N(305,12^2)$

(ii) $½X-20 \sim N(½\times100-20, ½^2\times4^2)=N(30,2^2)$

(iii) $¼(X-100) \sim N\left(¼\times(100-100), ¼^2\times4^2\right)=N(0,1)$

We have transformed $N(100,4^2)$ into the standard normal distribution. This process is called ***standardising****. We shall be using it to calculate probabilities for any normal distribution.*

Solution 1.11

(i) $P(X>110)=P\left(Z>\frac{110-100}{\sqrt{16}}\right)$

$=P(Z>2.5)$

$=1-P(Z<2.5)$ using area $=1$

$=1-0.99379=0.00621$ from tables

(ii) Following the same steps:

$$P(X>87)=P\left(Z>\frac{87-100}{\sqrt{16}}\right)$$

$=P(Z>-3.25)$

$=P(Z<3.25)$ by symmetry

$=0.99942$ from tables

(iii) $P(95<X<107)=P(X<107)-P(X<95)$

Now:

$$P(X<107)=P\left(Z<\frac{107-100}{\sqrt{16}}\right) \quad \text{standardising}$$

$=P(Z<1.75)$

$=0.95994$ from tables

$$P(X<95)=P\left(Z<\frac{95-100}{\sqrt{16}}\right) \quad \text{standardising}$$

$=P(Z<-1.25)$

$=P(Z>1.25)$ by symmetry

$=1-P(Z<1.25)$ using area $=1$

$=1-0.89435$ using tables

$=0.10565$

Therefore:

$$P(95<X<107)=0.95994-0.10565=0.85429$$

Solution 1.12

Let X be the height of a male actuary, so we have $X \sim N(178, 250)$

(i) $$P(X < 186) = P\left(Z < \frac{186-178}{\sqrt{250}}\right) \quad \text{standardising}$$

$$= P(Z < 0.506) \quad \text{3 DP}$$

The tabulated probabilities either side of 0.506 are:

$$P(Z < 0.50) = 0.69146$$

$$P(Z < 0.51) = 0.69497$$

Using linear interpolation:

$$P(Z < 0.506) = 0.69146 + 0.6 \times (0.69497 - 0.69146) = 0.69357$$

Hence:

$$P(X < 186) = 0.69357$$

Note that we rounded the standardised value to 3 DP. We could have kept the complete figure of 0.5059644…, in which case we would have obtained:

$$P(X < 0.5059644\ldots) = 0.69146 + 0.59644\ldots \times (0.69497 - 0.69146)$$

$$= 0.69355$$

This does make a difference. We will give both the rounded and full solutions from now on.

(ii) $$P(X>160)=P\left(Z>\frac{160-178}{\sqrt{250}}\right) \quad \text{standardising}$$

$$=P(Z>-1.138) \quad \text{3 DP}$$

$$=P(Z<1.138) \quad \text{by symmetry}$$

The tabulated probabilities either side of 1.138 are:

$$P(Z<1.13)=0.87076$$

$$P(Z<1.14)=0.87286$$

Using linear interpolation:

$$P(Z<1.138)=0.87076+0.8\times(0.87286-0.87076)=0.87244$$

So:

$$P(X>160)=0.87244$$

Using the complete standardised figure would give an answer of 0.87253.

(iii) $P(150 < X < 190) = P(X < 190) - P(X < 150)$

$$= P\left(Z < \frac{190-178}{\sqrt{250}}\right) - P\left(Z < \frac{150-178}{\sqrt{250}}\right) \text{ standardising}$$

$$= P(Z < 0.759) - P(Z < -1.771)$$

$P(Z < 0.759)$ can be found directly from the standard normal table using interpolation. The tabulated probabilities either side of 0.759 are:

$$P(Z < 0.75) = 0.77337$$

$$P(Z < 0.76) = 0.77637$$

This gives:

$$P(Z < 0.759) = 0.77337 + 0.9 \times (0.77637 - 0.77337) = 0.77607$$

Now:

$$P(Z < -1.771) = P(Z > 1.771) \quad \text{by symmetry}$$

$$= 1 - P(Z < 1.771) \quad \text{using area} = 1$$

The tabulated probabilities either side of 1.771 are:

$$P(Z < 1.77) = 0.96164$$

$$P(Z < 1.78) = 0.96246$$

This gives:

$$P(Z < 1.771) = 0.96164 + 0.1 \times (0.96246 - 0.96164) = 0.96172$$

So:

$$P(Z < -1.771) = 1 - 0.96172 = 0.03828$$

Therefore:

$$P(150 < X < 190) = 0.77607 - 0.03828 = 0.73779$$

Using the complete standardised figures would give 0.73777.

Solution 1.13

Rewriting this probability as a range:

$$\begin{aligned} P(|X-5|<2) &= P(-2<X-5<2) \\ &= P(3<X<7) \\ &= P(X<7)-P(X<3) \end{aligned}$$

$$\begin{aligned} P(X<7) &= P\left(Z<\frac{7-6}{\sqrt{1}}\right) && \text{standardising} \\ &= P(Z<1) && \text{from tables} \\ &= 0.84134 \end{aligned}$$

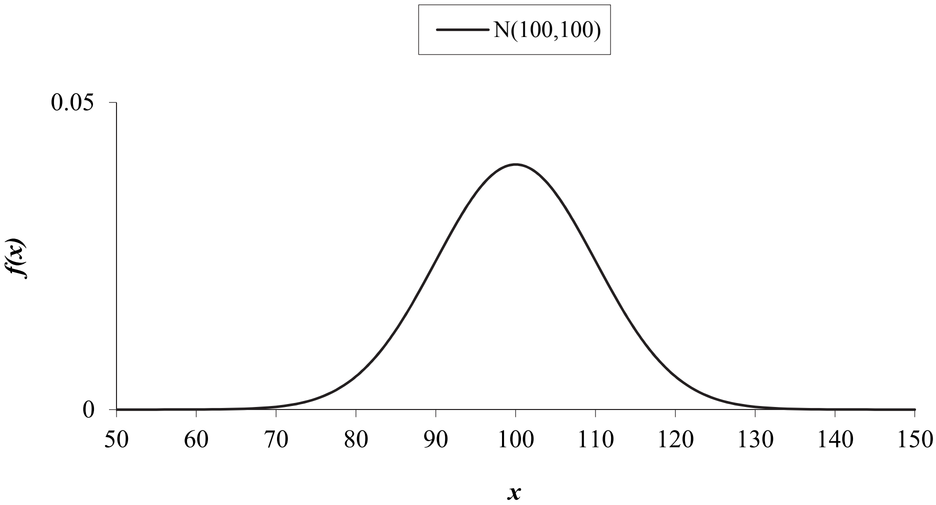
$$\begin{aligned} P(X<3) &= P\left(Z<\frac{3-6}{\sqrt{1}}\right) && \text{standardising} \\ &= P(Z<-3) \\ &= P(Z>3) && \text{by symmetry} \\ &= 1-P(Z<3) && \text{using area}=1 \\ &= 1-0.99865 && \text{from tables} \\ &= 0.00135 \end{aligned}$$

Hence:

$$P(|X-5|<2) = 0.84134-0.00135 = 0.83999$$

Solutions to extra practice questions

P1.1 The graph for $X \sim N(100,100)$ is:



The majority of the values are between $(100-3\times 10, 100+3\times 10)=(70,130)$.

The height can be found from $f(100)=\dfrac{1}{\sqrt{2\pi 100}}e^0=0.040$.

P1.2 (i)

$$\begin{aligned} P(Z>1.6) &= 1-P(Z<1.6) && \text{using area}=1 \\ &= 1-0.94520 && \text{from tables} \\ &= 0.05480 \end{aligned}$$

(ii)

$$P(Z<2.84)=0.99774 \qquad \text{from tables}$$

(iii)

$$\begin{aligned} P(Z<-0.42) &= P(Z>0.42) && \text{by symmetry} \\ &= 1-P(Z<0.42) && \text{using area}=1 \\ &= 1-0.66276 && \text{from tables} \\ &= 0.33724 \end{aligned}$$

(iv)

$$\begin{aligned} P(Z>-2.61) &= P(Z<2.61) && \text{by symmetry} \\ &= 0.99547 && \text{from tables} \end{aligned}$$

(v) $P(-1.69 < Z < 0.52) = P(Z < 0.52) - P(Z < -1.69)$

Now:

$$P(Z < 0.52) = 0.69847 \qquad \text{from tables}$$

$$\begin{aligned} P(Z < -1.69) &= P(Z > 1.69) && \text{by symmetry} \\ &= 1 - P(Z < 1.69) && \text{using area} = 1 \\ &= 1 - 0.95449 && \text{from tables} \\ &= 0.04551 \end{aligned}$$

Hence:

$$P(-1.69 < Z < 0.52) = 0.69847 - 0.04551 = 0.65296$$

(vi) $P(-3.05 < Z < -1.7) = P(Z < -1.7) - P(Z < -3.05)$

Now:

$$\begin{aligned} P(Z < -1.7) &= P(Z > 1.7) && \text{by symmetry} \\ &= 1 - P(Z < 1.7) && \text{using area} = 1 \\ &= 1 - 0.95543 && \text{from tables} \\ &= 0.04457 \end{aligned}$$

$$\begin{aligned} P(Z < -3.05) &= P(Z > 3.05) && \text{by symmetry} \\ &= 1 - P(Z < 3.05) && \text{using area} = 1 \\ &= 1 - 0.99886 && \text{from tables} \\ &= 0.00114 \end{aligned}$$

Hence:

$$P(-3.05 < Z < -1.7) = 0.04457 - 0.00114 = 0.04343$$

P1.3 (i) The tabulated probabilities either side of $P(X<0.371)$ are:

$$P(Z<0.37)=0.64431$$
$$P(Z<0.38)=0.64803$$

Using linear interpolation:

$$P(Z<0.371)=0.64431+0.1\times(0.64803-0.64431)=0.64468$$

(ii) $P(Z>2.598)=1-P(Z<2.598)$ using area $=1$

The tabulated probabilities either side of $P(Z<2.598)$ are:

$$P(Z<2.59)=0.99520$$
$$P(Z<2.60)=0.99534$$

Using linear interpolation:

$$P(Z<2.598)=0.99520+0.8\times(0.99534-0.99520)=0.99531$$

This gives:

$$P(Z>2.598)=1-0.99531=0.00469$$

(iii) $P(Z>-0.904)=P(Z<0.904)$ by symmetry

The tabulated probabilities either side of $P(Z<0.904)$ are:

$$P(Z<0.90)=0.81594$$
$$P(Z<0.91)=0.81859$$

Using linear interpolation:

$$P(Z<0.904)=0.81594+0.4\times(0.81859-0.81594)=0.81700$$

This gives:

$$P(Z>-0.904)=0.8170$$

(iv) $P(Z<-2.319)=P(Z>2.319)$ by symmetry

$=1-P(Z<2.319)$ using area $=1$

The tabulated probabilities either side of $P(Z<2.319)$ are:

$$P(Z<2.31)=0.98956$$

$$P(Z<2.32)=0.98983$$

Using liner interpolation:

$$P(Z<2.319)=0.98956+0.9\times(0.98983-0.98956)=0.98980$$

This gives:

$$P(Z<-2.319)=1-0.98980=0.01020$$

(v) $P(1.572<Z<3.087)=P(Z<3.087)-P(Z<1.572)$

For $P(Z<3.087)$ the tabulated probabilities either side are:

$$P(Z<3.08)=0.99896$$

$$P(Z<3.09)=0.99900$$

Using linear interpolation:

$$P(Z<3.087)=0.99896+0.7\times(0.99900-0.99896)=0.99899$$

For $P(Z<1.572)$ the tabulated probabilities either side are:

$$P(Z<1.57)=0.94179$$

$$P(Z<1.58)=0.94295$$

Using linear interpolation:

$$P(Z<1.572)=0.94179+0.2\times(0.94295-0.94179)=0.94202$$

Hence:

$$P(1.572 < Z < 3.087) = 0.99899 - 0.94202 = 0.05697$$

(vi) $P(-1.382 < Z < 0.493) = P(Z < 0.493) - P(Z < -1.382)$

For $P(Z < 0.493)$ the tabulated probabilities either side are:

$$P(Z < 0.49) = 0.68793$$

$$P(Z < 0.50) = 0.69146$$

Using linear interpolation:

$$P(Z < 0.493) = 0.68793 + 0.3 \times (0.69146 - 0.68793) = 0.68899$$

Now:

$$\begin{aligned} P(Z < -1.382) &= P(Z > 1.382) && \text{by symmetry} \\ &= 1 - P(Z < 1.382) && \text{using area} = 1 \end{aligned}$$

The tabulated probabilities either side of $P(Z < 1.382)$ are:

$$P(Z < 1.38) = 0.91621$$

$$P(Z < 1.39) = 0.91774$$

Using linear interpolation:

$$P(Z < 1.382) = 0.91621 + 0.2 \times (0.91774 - 0.91621) = 0.91652$$

This gives:

$$P(Z < -1.382) = 1 - 0.91652 = 0.08348$$

Hence:

$$P(-1.382 < Z < 0.493) = 0.68899 - 0.08348 = 0.60551$$

P1.4 (i) $P(X>117)=P\left(Z>\frac{117-128}{8}\right)$ standardising

$=P(Z>-1.375)$

$=P(Z<1.375)$ by symmetry

The tabulated probabilities either side of $P(Z<1.375)$ are:

$P(Z<1.37)=0.91466$

$P(Z<1.38)=0.91621$

Using linear interpolation (or just finding the average):

$P(Z<1.375)=0.91466+0.5\times(0.91621-0.91466)=0.91544$

Hence:

$P(X>117)=0.91544$

(ii) $P(Y<30)=P\left(Z<\frac{30-23}{5}\right)$ standardising

$=P(Z<1.4)$

$=0.91924$

(iii) $P(W>297)=P\left(Z>\frac{297-285}{\sqrt{34}}\right)$ standardising

$=P(Z>2.058)$ 3 DP

$=1-P(Z<2.058)$ using area $=1$

The tabulated probabilities either side of $P(Z<2.058)$ are:

$P(Z<2.05)=0.97982$

$P(Z<2.06)=0.98030$

Using linear interpolation:

$P(Z<2.058)=0.97982+0.8\times(0.98030-0.97982)=0.98020$

Hence:

$$P(W > 297) = 1 - 0.98020 = 0.01980$$

(iv) $P(V < 90) = P\left(Z < \frac{90-96}{\sqrt{12}}\right)$ standardising

$= P(Z < -1.732)$ 3 DP

$= P(Z > 1.732)$ by symmetry

$= 1 - P(Z < 1.732)$ using area $= 1$

The tabulated probabilities either side of $P(Z < 1.732)$ are:

$$P(Z < 1.73) = 0.95818$$

$$P(Z < 1.74) = 0.95907$$

Using linear interpolation:

$$P(Z < 1.732) = 0.95818 + 0.2 \times (0.95907 - 0.95818) = 0.95836$$

Hence:

$$P(V < 90) = 1 - 0.95836 = 0.04164$$

Using the complete standardised figure would give:

$$P(V < 90) = 1 - 0.95833 = 0.04167$$

(v) $P(3<T<8)=P(T<8)-P(T<3)$

$$=P\left(Z<\frac{8-5}{\sqrt{2}}\right)-P\left(Z<\frac{3-5}{\sqrt{2}}\right) \qquad \text{standardising}$$

$$=P(Z<2.121)-P(Z<-1.414) \qquad \text{3 DP}$$

For $P(Z<2.121)$ the tabulated probabilities either side are:

$$P(Z<2.12)=0.98300$$

$$P(Z<2.13)=0.98341$$

Using linear interpolation:

$$P(Z<2.121)=0.98300+0.1\times(0.98341-0.98300)=0.98304$$

Now:

$$P(Z<-1.414)=P(Z>1.414) \qquad \text{by symmetry}$$

$$=1-P(Z<1.414) \qquad \text{using area}=1$$

The tabulated probabilities either side of $P(Z<1.414)$ are:

$$P(Z<1.41)=0.92073$$

$$P(Z<1.42)=0.92220$$

Using linear interpolation:

$$P(Z<1.414)=0.92073+0.4\times(0.92220-0.92073)=0.92132$$

This gives:

$$P(Z<-1.414)=1-0.92132=0.07868$$

Therefore:

$$P(3<T<8)=0.98304-0.07868=0.90436$$

Using the complete standardised figures would have given:

$$P(3<T<8)=0.98305-0.07865=0.90440$$

P1.5 (i) Rewriting this probability as a range:

$$\begin{aligned}P(|X-17|<2) &= P(-2<X-17<2)\\ &= P(15<X<19)\\ &= P(X<19)-P(X<15)\\ &= P\left(Z<\frac{19-17}{\sqrt{5}}\right)-P\left(Z<\frac{15-17}{\sqrt{5}}\right) && \text{standardising}\\ &= P(Z<0.894)-P(Z<-0.894) && \text{3 DP}\end{aligned}$$

For $P(Z<0.894)$ the tabulated probabilities either side are:

$$P(Z<0.89)=0.81327$$
$$P(Z<0.90)=0.81594$$

Using linear interpolation:

$$P(Z<0.894)=0.81327+0.4\times(0.81594-0.81327)=0.81434$$

Now:

$$\begin{aligned}P(Z<-0.894) &= P(Z>0.894) && \text{by symmetry}\\ &= 1-P(Z<0.894) && \text{using area}=1\end{aligned}$$

Since we have just calculated $P(Z<0.894)$ we get:

$$P(Z<-0.894)=1-0.81434=0.18566$$

Therefore:

$$P(|X-17|<2)=0.81434-0.18566=0.62868$$

Using the complete figures for interpolation would give:

$$P(|X-17|<2)=0.81445-0.18555=0.62890$$

(ii) Using the conditional probability formula:

$$P(X<15\,|\,X<16)=\frac{P(X<15 \text{ and } X<16)}{P(X<16)}=\frac{P(X<15)}{P(X<16)}$$

$$\begin{aligned} P(X<15) &= P\left(Z<\frac{15-17}{\sqrt{5}}\right) && \text{standardising}\\ &= P(Z<-0.894) && \text{3 DP}\\ &= P(Z>0.894) && \text{by symmetry}\\ &= 1-P(Z<0.894) && \text{using area}=1\\ &= 0.18566 && \text{from part (i)} \end{aligned}$$

$$\begin{aligned} P(X<16) &= P\left(Z<\frac{16-17}{\sqrt{5}}\right) && \text{standardising}\\ &= P(Z<-0.447) && \text{3 DP}\\ &= P(Z>0.447) && \text{by symmetry}\\ &= 1-P(Z<0.447) && \text{using area}=1 \end{aligned}$$

The tabulated probabilities either side of $P(Z<0.447)$ are:

$$P(Z<0.44)=0.67003$$

$$P(Z<0.45)=0.67364$$

Using linear interpolation:

$$P(Z<0.447)=0.67003+0.7\times(0.67364-0.67003)=0.67256$$

This gives:

$$P(X<16)=1-0.67256=0.32744$$

Hence:

$$P(X<15\,|\,X<16)=\frac{0.18566}{0.32744}=0.5670$$

Using the complete standardised figures would give 0.5668.

P1.6 Let X be the sizes of claims, then $X \sim N(4\,000, 600^2)$.

We want:

$$P(X>4{,}000\,|\,X>3{,}400)=\frac{P(X>4{,}000 \text{ and } X>3{,}400)}{P(X>3{,}400)}=\frac{P(X>4{,}000)}{P(X>3{,}400)}$$

Now:

$$\begin{aligned} P(X>4{,}000) &= P\left(Z>\frac{4{,}000-4{,}000}{600}\right) && \text{standardising} \\ &= P(Z>0) \\ &= 1-P(Z<0) && \text{using area} = 1 \\ &= 1-0.5 && \text{from tables} \\ &= 0.5 \end{aligned}$$

$$\begin{aligned} P(X>3{,}400) &= P\left(Z>\frac{3{,}400-4{,}000}{600}\right) && \text{standardising} \\ &= P(Z>-1) \\ &= P(Z<1) && \text{by symmetry} \\ &= 0.84134 && \text{from tables} \end{aligned}$$

Therefore:

$$P(X>4{,}000\,|\,X>3{,}400)=\frac{0.5}{0.84134}=0.59429$$